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Time-domain Green's functions for unsaturated soils. Part II: Three-dimensional solution

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Abstract

The presented paper has been dedicated to complete the closed form *three-dimensional* fundamental solutions of the governing differential equations for an unsaturated deformable porous media with linear elastic behavior and a symmetric spherical domain in both Laplace transform and time domains. The governing differential equations consist of equilibrium, air and water transfer equations including the suction effect and dissolved air in water. The obtained Green's functions have been derived exactly, for the first time, using the linear form of the governing differential equations and considering the effects of non-linearity of the governing equations and have been verified in both frequency and time domains.

Keywords: Green's function; Unsaturated soil; Laplace transform; Boundary element method

1. Introduction

This paper is the second part of a pair of papers that attempt to derive the fundamental solutions for the governing differential equations of the unsaturated soils with elastic linear behavior for solid skeleton in symmetric spherical coordinates. In the first part, the closed form fundamental solutions in the *two-dimensional* case were presented in both frequency and time domains using the linear form of the governing differential equations and considering the effects of non-linearity of the governing

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equations. In the second part the corresponding Green's functions will be derived and verified for the *three-dimensional* case.

Hereafter, having the complete two and three-dimensional time-dependent fundamental solutions for the unsaturated soils, seems to enable us to model this phenomena with the boundary element method, that specially for the soils media, regarding its capability of modeling infinite boundaries as well as other advantages, is of great effectiveness and applicability.

2. Review of the governing equations

The governing differential equations for unsaturated porous media consist of equilibrium equations, constitutive equations of the solid skeleton, and continuity and transfer equations for air and water. These equations that have been derived in the previous paper, are written as follow.

2.1. Equilibrium and constitutive equations of the solid skeleton

Equilibrium equations based on the two independent parameters $(\sigma - p_a)$ and $(p_a - p_w)$, with elastic or linear behavior, considering stress–strain and strain–deformation relations, are

$$(\lambda + \mu)u_{j,ij} + \mu u_{i,jj} + (D_s - 1)p_{a,i} - D_s p_{w,i} + b_i = 0 \tag{1}$$

in which λ and μ are Lamé's coefficients of soil elasticity, D_s is the coefficient of deformations due to suction effect and u, σ, p_a and p_w stand for displacement of soil's solid skeleton, stress and air and water pressures, respectively. b denotes the body forces.

2.2. Continuity and transfer equations for air

The final air transfer equation consisting of generalized Darcy's law for air transfer, conservation law for air mass and air and water coefficients of permeability is

$$\begin{aligned}\frac{\rho_a K_a}{\gamma_a}\nabla^2 p_a + \frac{H\rho_a K_w}{\gamma_w}\nabla^2 p_w = {} & -\rho_a \beta \hat{u}_{i,i}(1-H)\frac{\partial}{\partial t}(p_a - p_w) \\ & + \rho_a[1 - (\alpha + \beta(\hat{p}_a - \hat{p}_w))(1-H)]\frac{\partial}{\partial t}(u_{i,i})\end{aligned} \tag{2}$$

where ρ_a and γ_a are air density and unit weight, γ_w denotes water unit weight and finally α and β are constants. K_a and K_w are air and water coefficients of permeability. Henry's coefficient, H, denotes the amount of dissolved air in water. Also t stands for time variable.

ρ_a and K_a are assumed constant in space and dispensing with variations of ρ_a in time. Also ∇^2 stands for the Laplacian operator and the hat sign $(\hat{\ })$ denotes that the parameter is assumed constant during the infinitesimal period ∂t.

2.3. Continuity and transfer equations for water

With the same procedure presented for air transfer, the final transfer equation for water, considering water velocity, water coefficient of permeability and mass conservation law, will be obtained as

$$\frac{\rho_w K_w}{\gamma_w}\nabla^2 p_w = \rho_w \beta \hat{u}_{i,i}\frac{\partial}{\partial t}(p_a - p_w) + \rho_w[\alpha + \beta(\hat{p}_a - \hat{p}_w)]\frac{\partial}{\partial t}(u_{i,i}) \tag{3}$$

where ρ_w denotes water density.

3. Laplace transform

Applying the Laplace transform to eliminate the time variable from the governing partial differential equations and solving the differential equations in Laplace transform domain, the following simplified equations will be resulted:

$$c_{11}\tilde{u}_{j,ij} + c_{12}\tilde{u}_{i,jj} + c_{13}\tilde{p}_{a,i} + c_{14}\tilde{p}_{w,i} + c_{15} = 0 \tag{4}$$

$$c_{21}\tilde{u}_{i,i} + c_{22}\tilde{p}_a + c_{23}\nabla^2\tilde{p}_a + c_{24}\tilde{p}_w + c_{25}\nabla^2\tilde{p}_w + c_{26} = 0 \tag{5}$$

$$c_{31}\tilde{u}_{i,i} + c_{32}\tilde{p}_a + c_{33}\nabla^2\tilde{p}_w + c_{34}\tilde{p}_w + c_{35} = 0, \quad i,j = \overline{1,3} \tag{6}$$

where the tilde denotes the variables in Laplace domain and the c_{ij} coefficients are as defined in paper part I.

4. Green's functions

Simplifying the differential Eqs. (4)–(6) in the following matrix form:

$$[C_{ij}] \times \vec{u} = \vec{f} \tag{7}$$

where $C_{ij} = c_{ij} \times d_{ij}$ in which d_{ij} are the differential operators and

$$\begin{aligned} \omega_i &= \tilde{u}_i, \quad i = \overline{1,3} \\ \omega_4 &= \tilde{p}_a \\ \omega_5 &= \tilde{p}_w \end{aligned} \tag{8}$$

and

$$\begin{aligned} f_i &= -\tilde{b}_i, \quad i = \overline{1,3} \\ f_4 &= -c_{26} \\ f_5 &= -c_{35} \end{aligned} \tag{9}$$

and implementing the Kupradze (Kupradze et al., 1979) or Hörmander's method (Hörmander, 1963) to derive the fundamental solutions $G = [\tilde{g}_{ij}]$, one can obtain the final differential equation to solve as

$$(D_1\nabla^{10} + D_2\nabla^8 + D_3\nabla^6)\varphi + \frac{1}{s}\delta(x) = 0 \tag{10}$$

where s is the Laplace transform parameter and $\nabla^{2n} = (\nabla^2)^n$ is n occurrence(s) of the Laplacian operator. The D_1, D_2 and D_3 parameters are defined as

$$\begin{aligned} D_1 &= c_{12}^2(c_{11} + c_{12})c_{23}c_{33} \\ D_2 &= c_{12}^2(-c_{14}c_{23}c_{31} + c_{13}(c_{25}c_{31} - c_{21}c_{33}) - (c_{11} + c_{12})(c_{25}c_{32} - c_{22}c_{33} - c_{23}c_{34})) \\ D_3 &= c_{12}^2(c_{13}(c_{24}c_{31} - c_{21}c_{34}) + c_{14}(c_{21}c_{32} - c_{22}c_{31}) - (c_{11} + c_{12})(c_{24}c_{32} - c_{22}c_{34})). \end{aligned} \tag{11}$$

Executing the same procedure as *two-dimensional* case, one finds the λ_1 and λ_2 parameters as

$$\lambda_{1,2}^2 = \frac{-D_2 \pm \sqrt{D_2^2 - 4D_1D_3}}{2D_1} \tag{12}$$

and noting that Green's function of Helmholtz differential equation for an only r-dependent fully symmetric *three-dimensional* domain is (Arfken and Weber, 2001; Ocendon et al., 1999):

$$\Phi_i = \frac{e^{-\lambda_i r}}{4\pi r}, \quad i = \overline{1,2} \tag{13}$$

one can obtain:

$$\Phi = D_1 s \nabla^6(\varphi) = \frac{e^{-\lambda_2 r} - e^{-\lambda_1 r}}{4\pi r(\lambda_2^2 - \lambda_1^2)} \tag{14}$$

then by applying three times the following *three-dimensional* inverse Laplacian operator (Spiegel, 1999):

$$\nabla^{-2}(\vartheta) = \int_r \left(r^{-2} \int_r (r^2 \vartheta)\, dr \right) dr \tag{15}$$

the φ function will be obtained as

$$\varphi(r,s) = \frac{1}{4\pi r D_1 s(\lambda_2^2 - \lambda_1^2)} \left(\frac{e^{-\lambda_2 r}}{\lambda_2^6} - \frac{e^{-\lambda_1 r}}{\lambda_1^6} \right) \tag{16}$$

the $[\tilde{g}_{ij}]$ Green's functions or cofactor matrix components $[C^*_{ij}]$ are

$$\begin{aligned}
\tilde{g}_{ij} &= [\delta_{ij}(F_{11}\nabla^8 + F_{12}\nabla^6 + F_{13}\nabla^4) + (F_{21}\nabla^6\partial_i\partial_j + F_{22}\nabla^4\partial_i\partial_j + F_{23}\nabla^2\partial_i\partial_j)]\varphi \\
\tilde{g}_{i4} &= (F_{31}\nabla^6\partial_i + F_{32}\nabla^4\partial_i)\varphi \\
\tilde{g}_{i5} &= (F_{41}\nabla^6\partial_i + F_{42}\nabla^4\partial_i)\varphi \\
\tilde{g}_{4i} &= (F_{51}\nabla^6\partial_i + F_{52}\nabla^4\partial_i)\varphi \\
\tilde{g}_{5i} &= (F_{61}\nabla^6\partial_i + F_{62}\nabla^4\partial_i)\varphi \\
\tilde{g}_{44} &= (F_{71}\nabla^8 + F_{72}\nabla^6)\varphi \\
\tilde{g}_{45} &= (F_{73}\nabla^8 + F_{74}\nabla^6)\varphi \\
\tilde{g}_{54} &= (F_{75}\nabla^6)\varphi \\
\tilde{g}_{55} &= (F_{76}\nabla^8 + F_{77}\nabla^6)\varphi, \quad i,j = \overline{1,3}
\end{aligned} \tag{17}$$

where δ_{ij} is the Kronecker delta operator. The F_{ij} coefficients are presented in Appendix A.

4.1. Green's functions in Laplace transform domain

Substituting the φ function from Eqs. (16) and (17) and defining the Γ_i intermediate functions:

$$\begin{aligned}
\Gamma_1 &= K_{11}\Omega_{11} + K_{12}\Omega_{12} + K_{13}\Omega_{13} \\
\Gamma_2 &= K_{21}\Omega_{31} + K_{22}\Omega_{32} + K_{23}\Omega_{33} \\
\Gamma_3 &= K_{21}\Omega_{11} + K_{22}\Omega_{12} + K_{23}\Omega_{13}
\end{aligned} \tag{18}$$

the Green's functions in Laplace transform domain are as

$$\begin{aligned}
\tilde{g}_{ij} &= \frac{\delta_{ij}}{r}\Gamma_1 + \frac{1}{r^5}(3x_i x_j - \delta_{ij} r^2)\Gamma_2 + \frac{x_i x_j}{r^3}\Gamma_3 \\
\tilde{g}_{i4} &= -\frac{x_i}{r^3}(K_{31}\Omega_{31} + K_{32}\Omega_{32}) \\
\tilde{g}_{i5} &= -\frac{x_i}{r^3}(K_{41}\Omega_{31} + K_{42}\Omega_{32}) \\
\tilde{g}_{4i} &= -\frac{x_i}{r^3}(K_{51}\Omega_{21} + K_{52}\Omega_{22})
\end{aligned}$$

$$
\begin{aligned}
\tilde{g}_{5i} &= -\frac{x_i}{r^3}(K_{61}\Omega_{21} + K_{62}\Omega_{22}) \\
\tilde{g}_{44} &= \frac{1}{r}(K_{71}\Omega_{11} + K_{72}\Omega_{12}) \\
\tilde{g}_{45} &= \frac{1}{r}(K_{73}\Omega_{11} + K_{74}\Omega_{12}) \\
\tilde{g}_{54} &= \frac{1}{r}K_{75}\Omega_{12} \\
\tilde{g}_{55} &= \frac{1}{r}(K_{76}\Omega_{11} + K_{77}\Omega_{12}), \quad i,j = \overline{1,3}.
\end{aligned} \tag{19}
$$

The above Green's functions are also presented in extended form in Appendix D. From the relationships in Appendix D, one can see that $\tilde{g}_{4i} = s\tilde{g}_{i4}$ and $\tilde{g}_{5i} = s\tilde{g}_{i5}$ (Chen, 1994). The K_{ij} coefficients and the Ω_{ij} intermediate functions are shown in Appendices B and C, respectively.

4.2. Green's functions in the time domain

Applying the inverse Laplace transform to the Laplace transform domain Green's functions, requires finding the inverse Laplace transforms of the following terms:

$$
\begin{aligned}
&\frac{e^{-r\lambda_2}}{\lambda_2^2(\lambda_2^2-\lambda_1^2)}, \quad \frac{e^{-r\lambda_2}}{\lambda_2(\lambda_2^2-\lambda_1^2)}, \quad \frac{e^{-r\lambda_2}\lambda_2}{(\lambda_2^2-\lambda_1^2)}, \quad \frac{e^{-r\lambda_2}\lambda_2^2}{(\lambda_2^2-\lambda_1^2)}, \quad \frac{se^{-r\lambda_2}}{\lambda_2^4(\lambda_2^2-\lambda_1^2)}, \quad \frac{se^{-r\lambda_2}}{\lambda_2^3(\lambda_2^2-\lambda_1^2)}, \quad \frac{se^{-r\lambda_2}}{\lambda_2^2(\lambda_2^2-\lambda_1^2)}, \\
&\frac{se^{-r\lambda_2}}{\lambda_2(\lambda_2^2-\lambda_1^2)}, \quad \frac{e^{-r\lambda_2}}{s(\lambda_2^2-\lambda_1^2)}, \quad \frac{e^{-r\lambda_2}\lambda_2}{s(\lambda_2^2-\lambda_1^2)}, \quad \frac{e^{-r\lambda_2}\lambda_2^2}{s(\lambda_2^2-\lambda_1^2)}
\end{aligned} \tag{20}
$$

where

$$
\begin{aligned}
\lambda_1 &= \sqrt{m_1}\sqrt{s} \\
\lambda_2 &= \sqrt{m_2}\sqrt{s} \\
\lambda_2^2 - \lambda_1^2 &= m_3 s
\end{aligned} \tag{21}
$$

and the m_i coefficients in Eq. (21) are

$$
\begin{aligned}
m_{1,2} &= \frac{-\frac{D_2}{s} \pm \sqrt{\frac{D_2^2 - 4D_1D_3}{s^2}}}{2D_1} \\
m_3 &= m_2 - m_1.
\end{aligned} \tag{22}
$$

Referring to the Laplace transform tables, we have the inverse Laplace transforms of the following terms (Abramowitz and Stegun, 1965; Spiegel, 1965):

$$
\frac{e^{r\sqrt{s}}}{s}, \quad \frac{e^{r\sqrt{s}}}{s^2}, \quad \frac{e^{r\sqrt{s}}}{\sqrt{s}}, \quad \frac{e^{r\sqrt{s}}}{s\sqrt{s}}. \tag{23}
$$

The inverse Laplace transforms of the terms in Eq. (23) are shown as $\Lambda_{ij}[a,t]$ in Appendix E. Now, by applying the inverse Laplace transforms $\Lambda_{ij}[a,t]$, we can obtain the inverse Laplace transforms of the Green's functions in Eq. (19). For this purpose, the intermediate functions $\Psi_{ij}[r,t]$ are defined in Appendix F. Using the K_{ij} coefficients and the intermediate functions $\Psi_{ij}[r,t]$, we are able to derive the Green's functions in the time domain that are shown in Eq. (25). By defining Θ_i intermediate functions as

$$\begin{aligned}
\Theta_1 &= K_{11}\Psi_{11}[r,t] + K_{12}\Psi_{12}[r,t] + K_{13}\Psi_{13}[r,t] \\
\Theta_2 &= K_{21}\Psi_{31}[r,t] + K_{22}\Psi_{32}[r,t] + K_{23}\Psi_{33}[r,t] \\
\Theta_3 &= K_{21}\Psi_{11}[r,t] + K_{22}\Psi_{12}[r,t] + K_{23}\Psi_{13}[r,t]
\end{aligned} \tag{24}$$

the time-domain Green's functions are

$$\begin{aligned}
g_{ij}[r,x_i,x_j,t] &= \frac{\delta_{ij}}{r}\Theta_1 + \frac{1}{r^5}(3x_ix_j - \delta_{ij}r^2)\Theta_2 + \frac{x_ix_j}{r^3}\Theta_3 \\
g_{i4}[r,x_i,t] &= -\frac{x_i}{r^3}(K_{31}\Psi_{31}[r,t] + K_{32}\Psi_{32}[r,t]) \\
g_{i5}[r,x_i,t] &= -\frac{x_i}{r^3}(K_{41}\Psi_{31}[r,t] + K_{42}\Psi_{32}[r,t]) \\
g_{4i}[r,x_i,t] &= -\frac{x_i}{r^3}(K_{51}\Psi_{21}[r,t] + K_{52}\Psi_{22}[r,t]) \\
g_{5i}[r,x_i,t] &= -\frac{x_i}{r^3}(K_{61}\Psi_{21}[r,t] + K_{62}\Psi_{22}[r,t]) \\
g_{44}[r,t] &= \frac{1}{r}(K_{71}\Psi_{11}[r,t] + K_{72}\Psi_{12}[r,t]) \\
g_{45}[r,t] &= \frac{1}{r}(K_{73}\Psi_{11}[r,t] + K_{74}\Psi_{12}[r,t]) \\
g_{54}[r,t] &= \frac{1}{r}K_{75}\Psi_{12}[r,t] \\
g_{55}[r,t] &= \frac{1}{r}(K_{76}\Psi_{11}[r,t] + K_{77}\Psi_{12}[r,t]), \quad i,j = \overline{1,3}.
\end{aligned} \tag{25}$$

5. Verification

Since the solutions are being introduced for the first time and due to the lack of enough references, verification and comparison with other corresponding data is not possible. Again same as in the case of the *two-dimensional* solution, for the solutions (mathematical model) to be verified mathematically, we can show for example if the conditions approach to the poroelastostatic case, the corresponding Green's functions will approach to the poroelastostatic Green's functions {neglecting dissolved air in water and the suction effect (i.e. $H = D_s = 0$)}. Considering the Eqs. (4)–(6), the coefficients of terms with time variations or $\widehat{S}_r$ and $\hat{n}$ will be substituted with zero. This equals to substituting the terms ξ (or $\widehat{S}_r$) and η (or $(1-\widehat{S}_r)$) and also $\hat{u}_{i,i}$ in K_{ij} statements with zero. Therefore the only non-vanishing coefficients are

$$\begin{aligned}
K_{11} &= \frac{1}{4\pi\mu} \\
K_{21} &= -\frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)} \\
K_{31} &= -\frac{\gamma_a}{4\pi(\lambda+2\mu)K_a\rho_a} \\
K_{71} &= -\frac{\gamma_a}{4\pi K_a\rho_a} \\
K_{76} &= -\frac{\gamma_w}{4\pi K_w\rho_w}.
\end{aligned} \tag{26}$$

Among the Ω_{ij} terms in the Laplace transform Green's functions in Appendix C, the nonvanishing ones are

$$\begin{aligned} \Omega_{11} &= \frac{1}{s(\lambda_2^2 - \lambda_1^2)}\left(e^{-r\lambda_2}\lambda_2^2 - e^{-r\lambda_1}\lambda_1^2\right) \\ \Omega_{31} &= \frac{1}{s(\lambda_2^2 - \lambda_1^2)}\left(e^{-r\lambda_2}(1 + r\lambda_2) - e^{-r\lambda_1}(1 + r\lambda_1)\right). \end{aligned} \tag{27}$$

By substituting the terms ξ (or $\widehat{S}_r$) and also $\hat{u}_{i,i}$ with zero, all the m_i terms and subsequently λ_1 and λ_2 will vanish. Therefore we have to evaluate the limits of Ω_{11} and Ω_{31} while λ_1 and λ_2 approach to zero:

$$\begin{aligned} \lim_{\lambda_1,\lambda_2\to 0}\{\Omega_{11}\} &= \frac{1}{s} \\ \lim_{\lambda_1,\lambda_2\to 0}\{\Omega_{31}\} &= -\frac{r^2}{2s}. \end{aligned} \tag{28}$$

In addition, while it seems to be normal, all of the Ω_{ij} terms in the Green's functions in Laplace transform domain that have zero coefficients, have no limits.

After some simplifications and using the above limits, the Green's functions in Laplace transform domain will be obtained as

$$\begin{aligned} \tilde{g}_{ij} &= \frac{(\lambda + 3\mu)r^2\delta_{ij} + (\lambda + \mu)x_i x_j}{8\pi r^3 s\mu(\lambda + 2\mu)} \\ \tilde{g}_{4i} &= \tilde{g}_{5i} = 0 \\ \tilde{g}_{i4} &= -\frac{\gamma_a x_i}{8\pi r s(\lambda + 2\mu)K_a\rho_a} \\ \tilde{g}_{i5} &= 0 \\ \tilde{g}_{44} &= -\frac{\gamma_a}{4\pi r s K_a \rho_a} \\ \tilde{g}_{45} &= \tilde{g}_{54} = 0 \\ \tilde{g}_{55} &= -\frac{\gamma_w}{4\pi r s K_w \rho_w}, \quad i,j = \overline{1,3} \end{aligned} \tag{29}$$

that their corresponding terms in time domain are

$$\begin{aligned} g_{ij} &= \frac{(\lambda + 3\mu)r^2\delta_{ij} + (\lambda + \mu)x_i x_j}{8\pi r^3 \mu(\lambda + 2\mu)} \\ g_{4i} &= g_{5i} = 0 \\ g_{i4} &= -\frac{\gamma_a x_i}{8\pi r(\lambda + 2\mu)K_a\rho_a} \\ g_{i5} &= 0 \\ g_{44} &= -\frac{\gamma_a}{4\pi r K_a \rho_a} \\ g_{45} &= g_{45} = 0 \\ g_{55} &= -\frac{\gamma_w}{4\pi r K_w \rho_w}, \quad i,j = \overline{1,3} \end{aligned} \tag{30}$$

that are exactly the poroelastostatic Green's functions (Banerjee, 1994; Gatmiri and Jabbari, 2004).

Furthermore, since

$$\Psi_{ij} = f(r^0), \quad i,j = \overline{1,3} \tag{31}$$

it may be concluded that the forms of the Green's functions from mathematical point of view and in terms of r are

$$\begin{aligned} & g_{ij} = f(r^{-3}, r^{-1}), \quad i,j = \overline{1,3} \\ & g_{i4}, g_{i5}, g_{4i}, g_{5i} = f(r^{-2}) \\ & g_{44}, g_{45}, g_{54}, g_{55} = f(r^{-1}) \end{aligned} \tag{32}$$

and all of these terms have definite limits (that approach to zero) when $r \to \infty$, and their singularity is only at $r = 0$.

6. Conclusion

In this research the closed form *three-dimensional* quasistatic Green's functions of the governing differential equations of unsaturated soils, including equilibrium equations with linear elastic constitutive equations and two equations of air and water transfer have been derived in both frequency and time domains, for the first time. The Green's functions are verified demonstrating that if the conditions approach to poroelastostatic case, the Green's functions will approach to poroelastostatic Green's functions exactly.

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Appendix A

F_{ij} coefficients:

$$F_{11} = c_{12}(c_{11} + c_{12})c_{23}c_{33}$$
$$F_{12} = c_{12}(-c_{14}c_{23}c_{31} + c_{13}(c_{25}c_{31} - c_{21}c_{33}) - (c_{11} + c_{12})(c_{25}c_{32} - c_{22}c_{33} - c_{23}c_{34}))$$
$$F_{13} = c_{12}(c_{14}(c_{21}c_{32} - c_{22}c_{31}) + c_{13}(c_{24}c_{31} - c_{21}c_{34}) - (c_{11} + c_{12})(c_{24}c_{32} - c_{22}c_{34}))$$
$$F_{21} = -c_{11}c_{12}c_{23}c_{33}$$
$$F_{22} = c_{12}(c_{14}c_{23}c_{31} + c_{13}(c_{21}c_{33} - c_{25}c_{31}) + c_{11}(c_{25}c_{32} - c_{22}c_{33} - c_{23}c_{34}))$$
$$F_{23} = c_{12}(c_{14}(c_{22}c_{31} - c_{21}c_{32}) + c_{13}(c_{21}c_{34} - c_{24}c_{31}) + c_{11}(c_{24}c_{32} - c_{22}c_{34}))$$
$$F_{31} = -c_{12}^2 c_{13}c_{33}, \qquad F_{32} = c_{12}^2(c_{14}c_{32} - c_{13}c_{34})$$
$$F_{41} = c_{12}^2(c_{13}c_{25} - c_{14}c_{23}), \qquad F_{42} = c_{12}^2(c_{13}c_{24} - c_{14}c_{22})$$
$$F_{51} = c_{12}^2(c_{25}c_{31} - c_{21}c_{33}), \qquad F_{52} = c_{12}^2(c_{24}c_{31} - c_{21}c_{34})$$
$$F_{61} = -c_{12}^2 c_{23}c_{31}, \qquad F_{62} = c_{12}^2(c_{21}c_{32} - c_{22}c_{31})$$
$$F_{71} = c_{12}^2(c_{11} + c_{12})c_{33}, \qquad F_{72} = c_{12}^2(-c_{14}c_{31} + (c_{11} + c_{12})c_{34})$$
$$F_{73} = -c_{12}^2(c_{11} + c_{12})c_{25}, \qquad F_{74} = -c_{12}^2(-c_{14}c_{21} + (c_{11} + c_{12})c_{24})$$
$$F_{75} = -c_{12}^2(-c_{13}c_{31} + (c_{11} + c_{12})c_{32}), \qquad F_{76} = c_{12}^2(c_{11} + c_{12})c_{23}$$
$$F_{77} = c_{12}^2(-c_{13}c_{21} + (c_{11} + c_{12})c_{22})$$

Appendix B

K_{ij} coefficients:

$$\xi = \alpha + \beta(\hat{p}_\mathrm{a} - \hat{p}_\mathrm{w}), \quad \eta = 1 - \xi(1 - H)$$

$$K_{11} = \frac{F_{11}}{4\pi D_1} = \frac{1}{4\pi\mu}$$

$$K_{12} = \frac{F_{12}}{4\pi D_1 s} = \frac{\beta(\lambda + 2\mu)(K_\mathrm{a}\gamma_\mathrm{w} + K_\mathrm{w}\gamma_\mathrm{a})\hat{u}_{i,i} + K_\mathrm{w}\gamma_\mathrm{a}(1-\xi)(-1+D_\mathrm{s}) - K_\mathrm{a}\gamma_\mathrm{w}\xi D_\mathrm{s}}{4\pi\mu(\lambda+2\mu)K_\mathrm{a}K_\mathrm{w}}$$

$$K_{13} = \frac{F_{13}}{4\pi D_1 s^2} = -\frac{\beta\gamma_\mathrm{a}\gamma_\mathrm{w}\hat{u}_{i,i}}{4\pi\mu(\lambda+2\mu)K_\mathrm{a}K_\mathrm{w}}$$

$$K_{21} = \frac{F_{21}}{4\pi D_1} = -\frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)}$$

$$K_{22} = \frac{F_{22}}{4\pi D_1 s} = -\frac{\beta(\lambda + \mu)(K_\mathrm{a}\gamma_\mathrm{w} + K_\mathrm{w}\gamma_\mathrm{a})\hat{u}_{i,i} + K_\mathrm{w}\gamma_\mathrm{a}(1-\xi)(-1+D_\mathrm{s}) - K_\mathrm{a}\gamma_\mathrm{w}\xi D_\mathrm{s}}{4\pi\mu(\lambda+2\mu)K_\mathrm{a}K_\mathrm{w}}$$

$$K_{23} = \frac{F_{23}}{4\pi D_1 s^2} = -K_{13}$$

$$K_{31} = \frac{F_{31}}{4\pi D_1} = \frac{\gamma_\mathrm{a}(-1+D_\mathrm{s})}{4\pi(\lambda+2\mu)K_\mathrm{a}\rho_\mathrm{a}}, \qquad K_{32} = \frac{F_{32}}{4\pi D_1 s} = -\frac{\beta\gamma_\mathrm{a}\gamma_\mathrm{w}\hat{u}_{i,i}}{4\pi(\lambda+2\mu)K_\mathrm{a}K_\mathrm{w}\rho_\mathrm{a}}$$

$$K_{41} = \frac{F_{41}}{4\pi D_1} = -\frac{H(-1+D_\mathrm{s})K_\mathrm{w}\gamma_\mathrm{a} + D_\mathrm{s}K_\mathrm{a}\gamma_\mathrm{w}}{4\pi(\lambda+2\mu)K_\mathrm{a}K_\mathrm{w}\rho_\mathrm{w}}, \qquad K_{42} = \frac{F_{42}}{4\pi D_1 s} = \frac{(-1+H)\beta\gamma_\mathrm{a}\gamma_\mathrm{w}\hat{u}_{i,i}}{4\pi(\lambda+2\mu)K_\mathrm{a}K_\mathrm{w}\rho_\mathrm{w}}$$

$$K_{51} = \frac{F_{51}}{4\pi D_1 s} = \frac{(1-\xi)\gamma_\mathrm{a}}{4\pi(\lambda+2\mu)K_\mathrm{a}}, \qquad K_{52} = \frac{F_{52}}{4\pi D_1 s^2} = \mu K_{23}$$

$$K_{61} = \frac{F_{61}}{4\pi D_1 s} = \frac{\gamma_\mathrm{w}\xi}{4\pi(\lambda+2\mu)K_\mathrm{w}}, \qquad K_{62} = \frac{F_{62}}{4\pi D_1 s^2} = K_{52}$$

$$K_{71} = \frac{F_{71}}{4\pi D_1} = -\frac{\gamma_\mathrm{a}}{4\pi K_\mathrm{a}\rho_\mathrm{a}}, \qquad K_{72} = \frac{F_{72}}{4\pi D_1 s} = \frac{\gamma_\mathrm{a}\gamma_\mathrm{w}(\xi D_\mathrm{s} - \beta(\lambda+2\mu)\hat{u}_{i,i})}{4\pi(\lambda+2\mu)K_\mathrm{a}K_\mathrm{w}\rho_\mathrm{a}}$$

$$K_{73} = \frac{F_{73}}{4\pi D_1} = \frac{H\gamma_\mathrm{a}}{4\pi K_\mathrm{a}\rho_\mathrm{w}}$$

$$K_{74} = \frac{F_{74}}{4\pi D_1 s} = -\frac{\gamma_\mathrm{a}\gamma_\mathrm{w}(\eta D_\mathrm{s} + (1-H)\beta\hat{u}_{i,i}(\lambda+2\mu))}{4\pi(\lambda+2\mu)K_\mathrm{a}K_\mathrm{w}\rho_\mathrm{w}}$$

$$K_{75} = \frac{F_{75}}{4\pi D_1 s} = -\frac{\gamma_\mathrm{a}\gamma_\mathrm{w}(\xi(1-D_\mathrm{s}) + \beta\hat{u}_{i,i}(\lambda+2\mu))}{4\pi(\lambda+2\mu)K_\mathrm{a}K_\mathrm{w}\rho_\mathrm{a}}, \qquad K_{76} = \frac{F_{76}}{4\pi D_1} = -\frac{\gamma_\mathrm{w}}{4\pi K_\mathrm{w}\rho_\mathrm{w}}$$

$$K_{77} = \frac{F_{77}}{4\pi D_1 s} = \frac{\gamma_\mathrm{a}\gamma_\mathrm{w}(\eta(1-D_\mathrm{s}) - (1-H)\beta\hat{u}_{i,i}(\lambda+2\mu))}{4\pi(\lambda+2\mu)K_\mathrm{a}K_\mathrm{w}\rho_\mathrm{w}}$$

Appendix C

The intermediate functions Ω_{ij}:

$$\Omega_{11} = \frac{\mathrm{e}^{-r\lambda_2}\lambda_2^2 - \mathrm{e}^{-r\lambda_1}\lambda_1^2}{s(\lambda_2^2 - \lambda_1^2)}, \qquad \Omega_{12} = \frac{\mathrm{e}^{-r\lambda_2} - \mathrm{e}^{-r\lambda_1}}{(\lambda_2^2 - \lambda_1^2)}$$

$$\Omega_{13} = \frac{s}{(\lambda_2^2 - \lambda_1^2)}\left(\frac{\mathrm{e}^{-r\lambda_2}}{\lambda_2^2} - \frac{\mathrm{e}^{-r\lambda_1}}{\lambda_1^2}\right), \quad \Omega_{21} = \frac{1}{(\lambda_2^2 - \lambda_1^2)}(\mathrm{e}^{-r\lambda_2}(1 + r\lambda_2) - \mathrm{e}^{-r\lambda_1}(1 + r\lambda_1))$$

$$\Omega_{22}=\frac{s}{(\lambda_2^2-\lambda_1^2)}\left(\frac{e^{-r\lambda_2}}{\lambda_2^2}(1+r\lambda_2)-\frac{e^{-r\lambda_1}}{\lambda_1^2}(1+r\lambda_1)\right),\quad \Omega_{31}=\frac{1}{s(\lambda_2^2-\lambda_1^2)}(e^{-r\lambda_2}(1+r\lambda_2)-e^{-r\lambda_1}(1+r\lambda_1))$$

$$\Omega_{32}=\frac{1}{(\lambda_2^2-\lambda_1^2)}\left(\frac{e^{-r\lambda_2}}{\lambda_2^2}(1+r\lambda_2)-\frac{e^{-r\lambda_1}}{\lambda_1^2}(1+r\lambda_1)\right),\quad \Omega_{33}=\frac{s}{(\lambda_2^2-\lambda_1^2)}\left(\frac{e^{-r\lambda_2}}{\lambda_2^4}(1+r\lambda_2)-\frac{e^{-r\lambda_1}}{\lambda_1^4}(1+r\lambda_1)\right)$$

Appendix D

The Green's functions in Laplace transform domain:

$$\begin{aligned}\tilde{g}_{ij}=&\,\delta_{ij}\left\{K_{11}\frac{(e^{-r\lambda_2}\lambda_2^2-e^{-r\lambda_1}\lambda_1^2)}{rs(\lambda_2^2-\lambda_1^2)}+K_{12}\frac{(e^{-r\lambda_2}-e^{-r\lambda_1})}{r(\lambda_2^2-\lambda_1^2)}+K_{13}\frac{s}{r(\lambda_2^2-\lambda_1^2)}\left(\frac{e^{-r\lambda_2}}{\lambda_2^2}-\frac{e^{-r\lambda_1}}{\lambda_1^2}\right)\right\}+K_{21}\\ &\times\frac{1}{r^5s(\lambda_2^2-\lambda_1^2)}\left[(3x_ix_j-\delta_{ij}r^2)(e^{-r\lambda_2}(1+r\lambda_2)-e^{-r\lambda_1}(1+r\lambda_1))+x_ix_jr^2(e^{-r\lambda_2}\lambda_2^2-e^{-r\lambda_1}\lambda_1^2)\right]+K_{22}\\ &\times\frac{1}{r^5(\lambda_2^2-\lambda_1^2)}\left[(3x_ix_j-\delta_{ij}r^2)\left(\frac{e^{-r\lambda_2}}{\lambda_2^2}(1+r\lambda_2)-\frac{e^{-r\lambda_1}}{\lambda_1^2}(1+r\lambda_1)\right)+x_ix_jr^2(e^{-r\lambda_2}-e^{-r\lambda_1})\right]+K_{23}\\ &\times\frac{s}{r^5(\lambda_2^2-\lambda_1^2)}\left[(3x_ix_j-\delta_{ij}r^2)\left(\frac{e^{-r\lambda_2}}{\lambda_2^4}(1+r\lambda_2)-\frac{e^{-r\lambda_1}}{\lambda_1^4}(1+r\lambda_1)\right)+x_ix_jr^2\left(\frac{e^{-r\lambda_2}}{\lambda_2^2}-\frac{e^{-r\lambda_1}}{\lambda_1^2}\right)\right]\end{aligned}$$

$$\tilde{g}_{i4}=-\frac{K_{31}x_i}{sr^3}\frac{1}{(\lambda_2^2-\lambda_1^2)}(e^{-r\lambda_2}(1+r\lambda_2)-e^{-r\lambda_1}(1+r\lambda_1))-\frac{K_{32}x_i}{r^3}\frac{1}{(\lambda_2^2-\lambda_1^2)}\left(\frac{e^{-r\lambda_2}}{\lambda_2^2}(1+r\lambda_2)-\frac{e^{-r\lambda_1}}{\lambda_1^2}(1+r\lambda_1)\right)$$

$$\tilde{g}_{i5}=-\frac{K_{41}x_i}{sr^3}\frac{1}{(\lambda_2^2-\lambda_1^2)}(e^{-r\lambda_2}(1+r\lambda_2)-e^{-r\lambda_1}(1+r\lambda_1))-\frac{K_{42}x_i}{r^3}\frac{1}{(\lambda_2^2-\lambda_1^2)}\left(\frac{e^{-r\lambda_2}}{\lambda_2^2}(1+r\lambda_2)-\frac{e^{-r\lambda_1}}{\lambda_1^2}(1+r\lambda_1)\right)$$

$$\tilde{g}_{4i}=-\frac{K_{51}x_i}{r^3}\frac{1}{(\lambda_2^2-\lambda_1^2)}(e^{-r\lambda_2}(1+r\lambda_2)-e^{-r\lambda_1}(1+r\lambda_1))-\frac{K_{52}x_i}{r^3}\frac{s}{(\lambda_2^2-\lambda_1^2)}\left(\frac{e^{-r\lambda_2}}{\lambda_2^2}(1+r\lambda_2)-\frac{e^{-r\lambda_1}}{\lambda_1^2}(1+r\lambda_1)\right)$$

$$\tilde{g}_{5i}=-\frac{K_{61}x_i}{r^3}\frac{1}{(\lambda_2^2-\lambda_1^2)}(e^{-r\lambda_2}(1+r\lambda_2)-e^{-r\lambda_1}(1+r\lambda_1))-\frac{K_{62}x_i}{r^3}\frac{s}{(\lambda_2^2-\lambda_1^2)}\left(\frac{e^{-r\lambda_2}}{\lambda_2^2}(1+r\lambda_2)-\frac{e^{-r\lambda_1}}{\lambda_1^2}(1+r\lambda_1)\right)$$

$$\tilde{g}_{44}=K_{71}\frac{(e^{-r\lambda_2}\lambda_2^2-e^{-r\lambda_1}\lambda_1^2)}{rs(\lambda_2^2-\lambda_1^2)}+K_{72}\frac{(e^{-r\lambda_2}-e^{-r\lambda_1})}{r(\lambda_2^2-\lambda_1^2)}$$

$$\tilde{g}_{45}=K_{73}\frac{(e^{-r\lambda_2}\lambda_2^2-e^{-r\lambda_1}\lambda_1^2)}{rs(\lambda_2^2-\lambda_1^2)}+K_{74}\frac{(e^{-r\lambda_2}-e^{-r\lambda_1})}{r(\lambda_2^2-\lambda_1^2)}$$

$$\tilde{g}_{54}=K_{75}\frac{(e^{-r\lambda_2}-e^{-r\lambda_1})}{r(\lambda_2^2-\lambda_1^2)}$$

$$\tilde{g}_{55}=K_{76}\frac{(e^{-r\lambda_2}\lambda_2^2-e^{-r\lambda_1}\lambda_1^2)}{rs(\lambda_2^2-\lambda_1^2)}+K_{77}\frac{(e^{-r\lambda_2}-e^{-r\lambda_1})}{r(\lambda_2^2-\lambda_1^2)}$$

Appendix E

The inverse Laplace transforms and intermediate functions $\Lambda_{ij}[a,t]$:

$$\mathrm{Erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \mathrm{e}^{-u^2}\,\mathrm{d}u$$

$$\Lambda_{11}[a,t] = \mathscr{L}^{-1}\left\{\frac{\mathrm{e}^{-a\sqrt{s}}}{s}\right\} = \mathrm{Erfc}\left(\frac{a}{2\sqrt{t}}\right)$$

$$\Lambda_{12}[a,t] = \mathscr{L}^{-1}\left\{\frac{\mathrm{e}^{-a\sqrt{s}}}{s^2}\right\} = \left(\frac{a^2}{2}+t\right)\mathrm{Erfc}\left(\frac{a}{2\sqrt{t}}\right) - a\sqrt{\frac{t}{\pi}}\mathrm{e}^{-\frac{a^2}{4t}}$$

$$\Lambda_{21}[a,t] = \mathscr{L}^{-1}\left\{\frac{\mathrm{e}^{-a\sqrt{s}}}{\sqrt{s}}\right\} = \frac{\mathrm{e}^{-\frac{a^2}{4t}}}{\sqrt{\pi t}}$$

$$\Lambda_{22}[a,t] = \mathscr{L}^{-1}\left\{\frac{\mathrm{e}^{-a\sqrt{s}}}{\sqrt{s^3}}\right\} = \sqrt{\frac{4t}{\pi}}\mathrm{e}^{-\frac{a^2}{4t}} - a\mathrm{Erfc}\left(\frac{a}{2\sqrt{t}}\right)$$

Appendix F

The intermediate functions $\Psi_{ij}[r,t]$:

$$\Psi_{11}[r,t] = \mathscr{L}^{-1}\{\Omega_{11}\} = \frac{1}{m_3}(m_2\Lambda_{11}[r\sqrt{m_2},t] - m_1\Lambda_{11}[r\sqrt{m_1},t])$$

$$\Psi_{12}[r,t] = \mathscr{L}^{-1}\{\Omega_{12}\} = \frac{1}{m_3}(\Lambda_{11}[r\sqrt{m_2},t] - \Lambda_{11}[r\sqrt{m_1},t])$$

$$\Psi_{13}[r,t] = \mathscr{L}^{-1}\{\Omega_{13}\} = \frac{1}{m_3}\left(\frac{1}{m_2}\Lambda_{11}[r\sqrt{m_2},t] - \frac{1}{m_1}\Lambda_{11}[r\sqrt{m_1},t]\right)$$

$$\Psi_{21}[r,t] = \mathscr{L}^{-1}\{\Omega_{21}\} = \frac{1}{m_3}(\Lambda_{11}[r\sqrt{m_2},t] - \Lambda_{11}[r\sqrt{m_1},t]) + \frac{r}{m_3}(\sqrt{m_2}\Lambda_{21}[r\sqrt{m_2},t] - \sqrt{m_1}\Lambda_{21}[r\sqrt{m_1},t])$$

$$\Psi_{22}[r,t] = \mathscr{L}^{-1}\{\Omega_{22}\} = \frac{1}{m_3}\left(\frac{1}{m_2}\Lambda_{11}[r\sqrt{m_2},t] - \frac{1}{m_1}\Lambda_{11}[r\sqrt{m_1},t]\right) + \frac{r}{m_3}\left(\frac{1}{\sqrt{m_2}}\Lambda_{21}[r\sqrt{m_2},t] - \frac{1}{\sqrt{m_1}}\Lambda_{21}[r\sqrt{m_1},t]\right)$$

$$\Psi_{31}[r,t] = \mathscr{L}^{-1}\{\Omega_{31}\} = \frac{1}{m_3}(\Lambda_{12}[r\sqrt{m_2},t] - \Lambda_{12}[r\sqrt{m_1},t]) + \frac{r}{m_3}(\sqrt{m_2}\Lambda_{22}[r\sqrt{m_2},t] - \sqrt{m_1}\Lambda_{22}[r\sqrt{m_1},t])$$

$$\Psi_{32}[r,t] = \mathscr{L}^{-1}\{\Omega_{32}\} = \frac{1}{m_3}\left(\frac{1}{m_2}\Lambda_{12}[r\sqrt{m_2},t] - \frac{1}{m_1}\Lambda_{12}[r\sqrt{m_1},t]\right) + \frac{r}{m_3}\left(\frac{1}{\sqrt{m_2}}\Lambda_{22}[r\sqrt{m_2},t] - \frac{1}{\sqrt{m_1}}\Lambda_{22}[r\sqrt{m_1},t]\right)$$

$$\Psi_{33}[r,t] = \mathscr{L}^{-1}\{\Omega_{33}\} = \frac{1}{m_3}\left(\frac{1}{m_2^2}\Lambda_{12}[r\sqrt{m_2},t] - \frac{1}{m_1^2}\Lambda_{12}[r\sqrt{m_1},t]\right) + \frac{r}{m_3}\left(\frac{1}{m_2\sqrt{m_2}}\Lambda_{22}[r\sqrt{m_2},t] - \frac{1}{m_1\sqrt{m_1}}\Lambda_{22}[r\sqrt{m_1},t]\right)$$

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